

Chapter 3 Theory of Angular Momentum

* Review: Continuous Symmetry in CM and QM, so far...

"Continuous Symmetry" [Operations]

← a family of "infinitesimal" $\left(\begin{array}{l} \text{CM: canonical transformations} \\ \text{QM: unitary transformations} \end{array} \right)$

⇒ "Lie" groups.

▽ Brief review of the canonical transformations in CM.
(see David Tong's lecture note, for more details)

def.

$$\left[\begin{array}{l} \dot{q}_i = \frac{\partial H}{\partial p_i} \\ \dot{p}_i = -\frac{\partial H}{\partial q_i} \end{array} \right] \xrightarrow{\text{Canonical transformation}} \left[\begin{array}{l} \dot{Q}_i = \frac{\partial H}{\partial P_i} \\ \dot{P}_i = -\frac{\partial H}{\partial Q_i} \end{array} \right]$$

$\left(\begin{array}{l} Q_i \equiv Q_i(\vec{q}, \vec{p}) \\ P_i \equiv P_i(\vec{q}, \vec{p}) \end{array} \right)$

→ Invariance of the Hamilton's EOM.

• In a more abstract form,

$$\vec{x} = (q_1, \dots, q_n, p_1, \dots, p_n)^T$$

$$\text{EOM: } \underline{\dot{\vec{x}}} = \mathcal{J} \frac{\partial H}{\partial \vec{x}} \quad \parallel \quad \mathcal{J} = \begin{pmatrix} 0 & \mathbf{I} \\ -\mathbf{I} & 0 \end{pmatrix}$$

• Transformations $q_i \rightarrow Q_i(\vec{q}, \vec{p})$, $p_i \rightarrow P_i(\vec{q}, \vec{p})$

can be now rewritten as

$$x_i \rightarrow y_i(\vec{x}) \quad \parallel \quad \vec{y} = (\vec{Q}, \vec{P})$$

• new EOM: $\ddot{y}_i = \frac{\partial y_i}{\partial x_j} \ddot{x}_j = \frac{\partial y_i}{\partial x_j} \mathcal{J}_{jk} \frac{\partial H}{\partial x_k}$

$$\Rightarrow \ddot{y}_i = \left(\frac{\partial y_i}{\partial x_j} \mathcal{J}_{jk} \frac{\partial y_k}{\partial x_k} \right) \frac{\partial H}{\partial y_k} \quad \Big| \quad = \frac{\partial H}{\partial y_k} \frac{\partial y_k}{\partial x_k}$$

$$\Rightarrow \ddot{\vec{y}} = \underbrace{\mathcal{J} \mathcal{J} \mathcal{J}^T}_{\text{in a matrix form}} \frac{\partial H}{\partial \vec{y}} \quad \parallel \mathcal{J}_{ij} = \frac{\partial \dot{q}_i}{\partial x_j}$$

Since the canonical transformations do not change the EOM: $\ddot{\vec{y}} = \mathcal{J} \frac{\partial H}{\partial \vec{y}}$.

\Rightarrow Requirement of the canonical transformations

$$: \quad \boxed{\mathcal{J} \mathcal{J} \mathcal{J}^T = \mathcal{J}} \quad *$$

$$\parallel [A, B]_{P.B.} = \frac{\partial A}{\partial q} \frac{\partial B}{\partial p} - \frac{\partial A}{\partial p} \frac{\partial B}{\partial q}.$$

\Rightarrow This is, in fact, equivalent to

$$[Q_i, Q_j]_{P.B.} = [P_i, P_j]_{P.B.} = 0$$

$$\text{and } [Q_i, P_j]_{P.B.} = \delta_{ij} \quad : \text{The invariance of the Poisson brackets.}$$

One can directly see this by using the def. of \mathcal{J}_{ij} ,

$$\mathcal{J}_{ij} = \frac{\partial \dot{q}_i}{\partial x_j} = \begin{pmatrix} \frac{\partial Q_i}{\partial q_j} & \frac{\partial Q_i}{\partial p_j} \\ \frac{\partial P_i}{\partial q_j} & \frac{\partial P_i}{\partial p_j} \end{pmatrix},$$

$$\text{and putting it into } \mathcal{J} \mathcal{J} \mathcal{J}^T = \mathcal{J}.$$

• Infinitesimal canonical transformations \Rightarrow

$$\begin{cases} q_i \rightarrow Q_i = q_i + \alpha F_i(\vec{q}, \vec{p}) \\ p_i \rightarrow P_i = p_i + \alpha E_i(\vec{q}, \vec{p}) \end{cases} \quad \text{for small } \alpha.$$

$$\Downarrow \mathcal{J} \mathcal{J} \mathcal{J}^T = \mathcal{J}$$

$$\begin{cases} q_i \rightarrow Q_i = q_i + \alpha \frac{\partial G}{\partial p_i} \\ p_i \rightarrow P_i = p_i - \alpha \frac{\partial G}{\partial q_i} \end{cases} \parallel G \equiv G(\vec{q}, \vec{p})$$

"Generating Function"

In other words, w.r.t. the infinitesimal α ,

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$$\left(\begin{array}{l} Q_i = q_i + \frac{dq_i}{d\alpha} \cdot \alpha \\ P_i = p_i + \frac{dp_i}{d\alpha} \cdot \alpha \end{array} \right) \Rightarrow \left(\begin{array}{l} \frac{dq_i}{d\alpha} = \frac{\partial G}{\partial p_i} \\ \frac{dp_i}{d\alpha} = -\frac{\partial G}{\partial q_i} \end{array} \right) !$$

• Infinitesimal change in an observable $A \equiv A(\vec{q}, \vec{p})$

$$\delta A = \frac{\partial A}{\partial q_i} \delta q_i + \frac{\partial A}{\partial p_i} \delta p_i$$

$$= \frac{\partial A}{\partial q_i} \cdot \alpha \frac{dq_i}{d\alpha} + \frac{\partial A}{\partial p_i} \cdot \alpha \frac{dp_i}{d\alpha}$$

$$= \alpha \frac{\partial A}{\partial q_i} \frac{\partial G}{\partial p_i} - \alpha \frac{\partial A}{\partial p_i} \frac{\partial G}{\partial q_i}$$

$$\Rightarrow \delta A = \alpha [A, G]_{P.B.} \quad \parallel \quad \stackrel{\text{def. } [A, B]_{P.B.}}{=} = \frac{\partial A}{\partial q} \frac{\partial B}{\partial p} - \frac{\partial A}{\partial p} \frac{\partial B}{\partial q}.$$

Ex. spatial translation

The Generator!

$$q_i \rightarrow Q_i = q_i + \alpha$$

$$p_i \rightarrow P_i = p_i$$

$$\Rightarrow G = \underline{P_i}$$

$$\text{and } \delta A = \alpha [A, P_i]_{P.B.} \quad (\delta A = A(q_i + \alpha) - A(q_i))$$

• Time-Evolution of an observable.

$$\frac{dA}{dt} = \frac{\partial A}{\partial q_i} \cdot \dot{q}_i + \frac{\partial A}{\partial p_i} \cdot \dot{p}_i + \frac{\partial A}{\partial t} \quad \dot{\vec{x}} = \mathcal{J} \frac{\partial H}{\partial \vec{x}} \quad (\text{EOM})$$

$$= \frac{\partial A}{\partial q_i} \frac{\partial H}{\partial p_i} - \frac{\partial A}{\partial p_i} \frac{\partial H}{\partial q_i} + \frac{\partial A}{\partial t}$$

$$\Rightarrow \underline{\frac{dA}{dt} = [A, H]_{P.B.}}$$

\longleftrightarrow Heisenberg EOM.

$$+ \frac{\partial A}{\partial t}$$

$$\left([,]_{PB} \rightarrow \frac{1}{i\hbar} [,] \right)$$

• Noether's theorem

a continuous symmetry \longrightarrow an integral of motion
(conserved quantity)

$$\dot{G} = \alpha [H, G]_{\text{p.B.}}$$

$$\oplus \frac{dG}{dt} = [G, H]_{\text{p.B.}} + \frac{\partial G}{\partial t}$$

\Rightarrow If H is invariant under a certain cont. sym. operation,
its generator is preserved.

ex. spatial translational invariance : $G = P$

$$\delta H = H(q+\alpha) - H(q) = 0$$

$$\rightarrow [H, P] = 0 \Rightarrow \frac{dP}{dt} = 0$$

linear momentum conservation.

ex. time translation invariance : $G = H$

$$\delta H = H(t+\alpha) - H(t) = 0$$

$$: [H, H] = 0, \quad \frac{\partial H}{\partial t} = 0 \quad (t\text{-indep. } H)$$

$$\Rightarrow \frac{dH}{dt} = 0 : \text{Energy conservation.}$$

Continuous symmetry in QM, so far ...

We have seen

- 1) spatial translation $\mathcal{T}(a) = \exp\left[-\frac{i}{\hbar} \tilde{P} a\right]$
- 2) time evolution

$$U(t) = \exp\left[-\frac{i}{\hbar} H t\right]$$

← The key property that we have used to derive these: \dagger

$$\underline{U(\alpha_1 + \alpha_2) = U(\alpha_1) \cdot U(\alpha_2)} \quad \parallel \begin{array}{l} \text{"Abelian"} \\ \text{group property.} \end{array} \quad \dots (*)$$

⊕

The Stone theorem

Given a set of unitary operators depending on a continuous parameter α and satisfying the Abelian group law, there exists a Hermitian operator G , called the infinitesimal generator of the transformation group $U(\alpha)$, such that $U(\alpha) = e^{i\alpha G}$.

- infinitesimal transformation can be written, $u = e^{-\frac{i}{\hbar} \alpha G}$
when (*) holds,
$$\underline{U(\alpha) \approx 1 - \frac{i}{\hbar} \alpha G + O[(\alpha/\hbar)^2]}$$

- The corresponding change of an observable:

$$\begin{aligned} F(a + \delta a) &= U^\dagger F(a) U \\ &= F(a) + \frac{i}{\hbar} \delta a [G, F] + \dots \end{aligned}$$

$$\Rightarrow \underline{\delta F = \frac{i}{\hbar} \delta a [G, F]} \quad \text{or} \quad \frac{\delta F}{\delta a} = \frac{i}{\hbar} [G, F]$$

ex. spatial translation.

$$\text{If } G = \tilde{P},$$

$$\textcircled{1} \quad \delta A = \alpha [A, \tilde{P}]_{PB} \iff \delta F = \frac{i}{\hbar} \delta a [\tilde{P}, F]$$

[classical]

[Quantum]

! "classical-quantum correspondence" !

$$\textcircled{2} \quad \lim_{\delta a \rightarrow 0} \frac{\delta F}{\delta a} = \frac{\partial F}{\partial x} = \frac{\hat{N}}{i\hbar} [\tilde{p}, F]$$

$$\Rightarrow [\tilde{p}, F(\tilde{x})] = -i\hbar \frac{\partial}{\partial \tilde{x}} F$$

$$\text{If } F(\tilde{x}) = \tilde{x}, \quad [\tilde{x}, \tilde{p}] = i\hbar.$$

ex. time-evolution.

$$\delta A = \delta t [A, H]_{\text{p.b.}} \longrightarrow \delta A = \frac{i}{\hbar} \delta t [H, A]$$

$$\Rightarrow \frac{dA}{dt} = \frac{1}{i\hbar} [A, H] : \text{Heisenberg EOM.}$$

Thus.

"G" is the same!

<u>Classical</u>	\longrightarrow	<u>Quantum</u>
Canonical transformation $\left(\begin{aligned} Q &= q + \alpha \frac{\partial G}{\partial p} \\ P &= p - \alpha \frac{\partial G}{\partial q} \end{aligned} \right)$		Unitary transformation. $\left(1 - \frac{i}{\hbar} \alpha G \right)$

$e^{-\frac{i}{\hbar} \alpha G}$
 \uparrow

(1) Rotations in C.M. and Q.M.

The trouble! : $U(\alpha_1) U(\alpha_2) \neq U(\alpha_2) U(\alpha_1)$
"non-Abelian"

$$e^{-\frac{i}{\hbar} \alpha_1 G_1} e^{-\frac{i}{\hbar} \alpha_2 G_2} = \exp \left[-\frac{i}{\hbar} (\alpha_1 G_1 + \alpha_2 G_2) \right]$$

only when $[G_1, G_2] = 0$.

* This is broken in general for Rotations.

(in Both of C.M. and Q.M.).

* (NOTE: We're talking about "3D" here.
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